Finite-Step Algorithms for Constructing Optimal CDMA Signature Sequences

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***Abstract*— A description of optimal sequences for direct-spread code division multiple access is a byproduct of recent characterizations of the sum capacity. This correspondence restates the sequence design problem as an inverse singular value problem and shows that the problem can be solved with finite-step algorithms from matrix theory. It proposes a new one-sided algorithm that is numerically stable and faster than previous methods.**

***Index Terms*— Algorithms, code division multiple access, inverse eigenvalue problems, optimal sequences, sum capacity**

I. Introduction

We consider the problem of designing signature sequences to maximize the sum capacity of a symbol-synchronous direct-spread code division multiple access (henceforth S-CDMA) system operating in the presence of white noise. This question has received a tremendous amount of attention in the information theory community over the last decade, e.g. [1]–[8]. These papers, however, could benefit from a matrix-theoretic perspective. First of all, they do not fully exploit the fact that sequence design is fundamentally an *inverse singular value problem* [9]. Second, finite-step algorithms to solve the sequence design problem have been available in the matrix computations literature for over two decades [10], [11]. Finally, researchers rarely mention computational complexity or numerical stability, which are both significant issues for any software.

This correspondence addresses sequence design using tools from matrix theory. Our approach clarifies and simplifies the treatment in comparison with existing information theory literature, and it also allows us to develop a new algorithm whose computational complexity is superior to earlier methods. In particular, this paper deals with the following issues.

1) We take advantage of the fact that the S-CDMA sequence design problem is equivalent with the classical Schur–Horn inverse eigenvalue problem. This perspective provides an efficient route to understanding the S-CDMA signature design literature. The power of this approach becomes clear when investigating more difficult design problems [12].
2) This connection leads us to several finite-step algorithms from matrix theory. We present numerically stable versions of these methods and study their computational complexity. Earlier authors were evidently unfamiliar with this work. For example, one of the algorithms in [2] seems to be identical with an algorithm published in 1983 [11].
3) Finally, we leverage our insights to develop a new finite-step algorithm for designing real S-CDMA signature sequences. This algorithm is numerically stable, and its time and storage complexity improve over all previous algorithms.

The S-CDMA signature design problem is usually studied in the real setting. In some related sequence design problems, however, the complex case is richer. Therefore, we have chosen to address the complex case instead; the real case follows from a transparent adaptation.

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II. Background

A. Synchronous DS-CDMA

Consider the uplink of an S-CDMA system with N users and a processing gain of d. Assume that $N > d$, since the analysis of the other case is straightforward. Assuming perfect synchronization, the equivalent baseband representation after matched filtering and sampling at the receiver is given by

$$\boldsymbol{y}[t] \;=\; \sum_{n=1}^{N} b_n[t]\, \boldsymbol{s}_n \;+\; \boldsymbol{v}[t]$$

where $\boldsymbol{y}[t] \in \mathbb{C}^d$ is the observation during symbol interval t, $\boldsymbol{s}_n \in \mathbb{C}^d$ is the signature of user n, $b_n[t] \in \mathbb{C}$ is the symbol transmitted by user n and $\boldsymbol{v}[t] \in \mathbb{C}^d$ is the realization of an independent and identically distributed complex Gaussian vector with zero mean and covariance matrix $\mathsf{\Sigma}$. We assume that the energy of each signature is normalized to unity, i.e. $\|\boldsymbol{s}_n\|_2 = 1$ for $n = 1, 2, \dots, N$. Define a $d \times N$ matrix whose columns are the signatures: $\mathsf{S} \stackrel{\text{def}}{=} \begin{bmatrix} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \dots & \boldsymbol{s}_N \end{bmatrix}$. Let S^* to denote the (conjugate) transpose of S. Note that $(\mathsf{S}^*\mathsf{S})_{nn} = 1$ for each $n = 1, \dots, N$. Assume that user n has an average power constraint

$$w_n \stackrel{\text{def}}{=} \frac{1}{T} \sum_{t=1}^{T} |b_n[t]|^2$$

where T is the number of symbol periods. Note that each w_n is strictly positive, and collect them in the diagonal matrix $\mathsf{W} \stackrel{\text{def}}{=} \operatorname{diag}(w_1, w_2, \dots, w_N)$. It is often more convenient to absorb the power constraints into the signatures, so we also define the weighted signature matrix $\mathsf{X} \stackrel{\text{def}}{=} \mathsf{S}\mathsf{W}^{1/2}$. Denote the n-th column of X as $\boldsymbol{x}_n$. For each n, one has the relationship

$$(\mathsf{X}^*\mathsf{X})_{nn} = \|\boldsymbol{x}_n\|_2^2 = w_n. \tag{1}$$

Viswanath and Anantharam have proven in [6] that, for real signatures, the sum capacity of the S-CDMA channel per degree of freedom is given by the expression

$$C_{\text{sum}} = \frac{1}{2d} \max_{\mathsf{S}} \log\det(\mathsf{I}_d + \mathsf{\Sigma}^{-1}\mathsf{S}\mathsf{W}\mathsf{S}^*). \tag{2}$$

(In the complex case, the sum capacity differs by a constant factor.) The basic sequence design problem is to produce a signature matrix S that solves the optimization problem (2). Three cases have been considered in the literature.

1) The white noise, equal power case was considered by Rupf and Massey in [1]. Here, the noise covariance matrix and the power constraint matrix are both multiples of the identity. That is, $\mathsf{\Sigma} = \sigma^2\,\mathsf{I}_d$, and $\mathsf{W} = w\,\mathsf{I}_N$.
2) Later, Viswanath and Anantharam addressed the situation of white noise and unequal user powers [2]. Here, the power constraints form a positive diagonal matrix W, and $\mathsf{\Sigma} = \sigma^2\,\mathsf{I}_d$.
3) Most recently, Viswanath and Anantharam have succeeded in characterizing the optimal sequences under colored noise and unequal user powers [6]. Here, $\mathsf{\Sigma}$ is an arbitrary positive semi-definite matrix, and W is a positive diagonal matrix.

We discuss each scenario in a subsequent section. The algorithms we develop can be used to construct optimal signatures for each case. Most previous work on sum capacity has not considered complex signature sequences. This note addresses the complex case exclusively because it subsumes the real case without any additional difficulty of argument.

B. A Sum Capacity Bound

In [1], Rupf and Massey produced an upper bound on the sum capacity under white noise with variance σ^2:

$$C_{\text{sum}} \le \tfrac{1}{2} \log \left(1 + \frac{\operatorname{Tr} W}{\sigma^2 \, d} \right) \tag{3}$$

where $\operatorname{Tr}(\cdot)$ indicates the trace operator. They also established a necessary and sufficient condition on the signatures for equality to be attained in the bound (3):

$$XX^* = SWS^* = \frac{\operatorname{Tr} W}{d} \, \mathrm{I}_d. \tag{4}$$

A matrix X that satisfies (4) is known as a *tight frame* [13] or a *general Welch-Bound-Equality sequence* (gWBE) [2]. As we shall see, a tight frame X does not exist for every choice of W. (A majorization condition must hold, as discussed in Section II-E.) A condition equivalent to (4) is that

$$X^* X = \frac{\operatorname{Tr} W}{d} \, P \tag{5}$$

where the matrix P represents an orthogonal projector from $\mathbb{C}^N$ onto a subspace of dimension d. Recall that an orthogonal projector is an idempotent, Hermitian matrix. That is, $P^2 = P$ and $P = P^*$. An orthogonal projector is also characterized as a Hermitian matrix whose nonzero eigenvalues are identically equal to one. In light of equation (1), the problem of constructing optimal signature sequences in the present setting is closely related to the problem of constructing an orthogonal projector with a specified diagonal.

C. White Noise, Equal Powers

Consider the case where the power constraints are equal, viz. $W = w \, \mathrm{I}_N$ for some positive number w. Then condition (4) for equality to hold in (3) becomes

$$w^{-1} \, XX^* = SS^* = \frac{N}{d} \, \mathrm{I}_d. \tag{6}$$

A matrix S which satisfies (6) is known as a *unit-norm tight frame* (UNTF) [13] or a *Welch-Bound-Equality sequence* (WBE) [1]. In fact, there always exist signature matrices that satisfy condition (6), and so the upper bound on the sum capacity can always be attained when the users' power constraints are equal [1]. The equation (6) can also be interpreted as a restriction on the singular values of the signature matrix. Under the assumptions of white noise and equal power constraints, a matrix S yields optimal signatures if and only if

1) each column of S has unit-norm and
2) the d nonzero singular values of S are identically equal to $\sqrt{N/d}$.

Therefore, this sequence design problem falls into the category of structured inverse singular value problems [9]. Note that condition 1) must hold irrespective of the type of noise.

D. Majorization

The bound (3) cannot be met for an arbitrary set of power constraints. The explanation requires a short detour. The k-th *order statistic* of a vector $\boldsymbol{v}$ is its k-th smallest entry, and it is denoted as $v_{(k)}$. Suppose that $\boldsymbol{w}$ and $\boldsymbol{\lambda}$ are N-dimensional, real vectors. Then $\boldsymbol{w}$ is said to *majorize* $\boldsymbol{\lambda}$ when their order statistics satisfy the following conditions:

$$\begin{aligned} \lambda_{(1)} &\le w_{(1)} \\ \lambda_{(1)} + \lambda_{(2)} &\le w_{(1)} + w_{(2)} \\ &\vdots \\ \lambda_{(1)} + \dots + \lambda_{(N-1)} &\le w_{(1)} + \dots + w_{(N-1)} \quad \text{and} \\ \lambda_{(1)} + \dots + \lambda_{(N)} &= w_{(1)} + \dots + w_{(N)}. \end{aligned} \tag{7}$$

The majorization relation (7) is commonly written as $\boldsymbol{w} \succcurlyeq \boldsymbol{\lambda}$ because it induces a partial ordering on $\mathbb{R}^N$. Note that the direction of the partial ordering is reversed in some treatments. An intuition which may help to clarify this definition is that the majorizing vector ($\boldsymbol{w}$) is an averaged version of the majorized vector ($\boldsymbol{\lambda}$); its components are clustered more closely together. It turns out that majorization defines the precise relationship between the diagonal entries of a Hermitian matrix and its spectrum.

Theorem 1 (Schur–Horn [14]): The diagonal entries of a Hermitian matrix majorize its eigenvalues. Conversely, if $\boldsymbol{w} \succcurlyeq \boldsymbol{\lambda}$, there exists a Hermitian matrix with diagonal elements listed by $\boldsymbol{w}$ and eigenvalues listed by $\boldsymbol{\lambda}$.

Schur demonstrated the necessity of the majorization condition in 1923, while Horn proved its sufficiency some thirty years later [14]. A comprehensive reference on majorization is [15].

E. White Noise, Unequal Powers

The Schur–Horn Theorem forbids the construction of an orthogonal projector with arbitrary diagonal entries. For this reason, (5) cannot always hold, and the upper bound (3) cannot always be attained.

The key result of [2] is a complete characterization of the sum capacity of the S-CDMA channel under white noise. Viswanath and Anantharam demonstrate that *oversized* users—those whose power constraints are too large relative to the others for the majorization condition to hold—must receive their own orthogonal channels to maximize the sum capacity of the system, and they provide a simple method of determining which users are oversized. The other users share the remaining dimensions equitably.

For reference, we include the Viswanath-Anantharam method for determining the set $\mathscr{K}$ of oversized users.

1) Initialize $\mathscr{K} = \emptyset$.
2) Terminate if $\sum_{n \notin \mathscr{K}} w_n \ge (d - |\mathscr{K}|) \max_{n \notin \mathscr{K}} w_n$.
3) Perform the update $\mathscr{K} \leftarrow \mathscr{K} \cup \arg\max_{n \notin \mathscr{K}} \{w_n\}$.
4) Return to Step 2.

Suppose that there are $m < d$ oversized users, whose signatures form the columns of S_0. Let the columns of S_1 list the signatures of the $(N - m)$ remaining users, and let the diagonal matrix W_1 list their power constraints. The conditions for achieving sum capacity follow.

1) The m oversized users receive orthogonal signatures: $S_0^* S_0 = \mathrm{I}_m$.
2) The remaining $(N - m)$ signatures are also orthogonal to the oversized users' signatures: $S_0^* S_1 = 0$.
3) The remaining users signatures satisfy

$$S_1 W_1 S_1^* = \frac{\operatorname{Tr} W_1}{d - m} \, \mathrm{I}_{d-m}.$$

Repeat the foregoing arguments to see that the sequence design problem still amounts to constructing a matrix with given column norms and singular spectrum. It is therefore an inverse singular value problem.

F. Total Squared Correlation

It is worth mentioning an equivalent formulation of the white-noise sequence design problem that provides a foundation for several iterative design algorithms [3]–[5], [7].

The *total weighted squared correlation* of a signature sequence is the quantity

$$\begin{aligned}\mathrm{TWSC}_W(S) &\stackrel{\text{def}}{=} \left\| W^{1/2} S^* S W^{1/2} \right\|_{\mathrm{F}}^2 = \left\| X^* X \right\|_{\mathrm{F}}^2 \\ &= \sum_{m,n=1}^{N} w_m\, w_n \, |\langle \boldsymbol{s}_m, \boldsymbol{s}_n \rangle|^2 .\end{aligned}$$

In a rough sense, this quantity measures how "spread out" the signature vectors are. Minimizing the TWSC of a signature sequence is the same as solving the optimization problem (2), as shown in [7]. A short algebraic manipulation shows that minimizing the TWSC is also equivalent to minimizing the quantity

$$\left\| X X^* - \frac{\operatorname{Tr} W}{d} \, \mathrm{I}_d \right\|_{\mathrm{F}}^2 .$$

In words, the singular values of an optimal weighted signature sequence X should be "as constant as possible." It should be emphasized that this equivalence only holds in the case of white noise.

G. Colored Noise, Unequal Powers

When the noise is colored, the situation is somewhat more complicated. Nevertheless, optimal sequence design still boils down to constructing a matrix with given column norms and singular spectrum. Viswanath and Anantharam show that the following procedure will solve the problem [6].

1) Compute an eigenvalue decomposition of the noise covariance matrix, $\Sigma = Q D Q^*$, where $D = \operatorname{diag} \boldsymbol{\sigma}$ for some non-negative vector $\boldsymbol{\sigma}$.
2) Use Algorithm $\mathcal{A}$ of [6] to determine $\boldsymbol{\mu}$, the Schur-minimal element of the set of possible eigenvalues of $S W S^* + \Sigma$.
3) Form the vector $\boldsymbol{\lambda} \stackrel{\text{def}}{=} \boldsymbol{\mu} - \boldsymbol{\sigma}$.
4) Compute an auxiliary signature matrix T with unit-norm columns so that $T W T^* = \operatorname{diag} \boldsymbol{\lambda}$.
5) The optimal signature matrix is $S \stackrel{\text{def}}{=} Q T$.

The computation in step (4) is equivalent to producing a $d \times N$ matrix $X \stackrel{\text{def}}{=} T W^{1/2}$. The columns of X must have squared norms listed by the diagonal of W. The vector $\boldsymbol{\lambda}$ must list the d nonzero squared singular values of X. This is another inverse singular value problem.

III. Constructing Unit-Norm Signature Sequences

Now that we have set out the conditions that an optimal signature sequence must satisfy, we may ask how to construct these sequences. It turns out that some useful algorithms have been available for a long time. But the connection with S-CDMA signature design has never been observed.

A positive semi-definite Hermitian matrix with a unit diagonal is also known as a *correlation matrix* [16]. We have seen that the Gram matrix $A \stackrel{\text{def}}{=} S^* S$ of an optimal signature matrix S is always a correlation matrix. Moreover, every correlation matrix with the appropriate spectrum can be factored to produce an optimal signature matrix [16]. Therefore, we begin with a basic technique for constructing correlation matrices with a preassigned spectrum.

A. A Numerically Stable, Finite Algorithm

In 1978, Bendel and Mickey presented an algorithm that uses a finite sequence of rotations to convert an arbitrary $N \times N$ Hermitian matrix with trace N into a unit-diagonal matrix that has the same spectrum [10]. We follow the superb exposition of Davies and Higham [16]. Brief discussions also appear on page 76 of Horn and Johnson [14] and in Problems 8.4.1 and 8.4.2 of Golub and van Loan [17].

Suppose that $A \in \mathbb{M}_N$ is a Hermitian matrix with $\operatorname{Tr} A = N$. (Let $\mathbb{M}_N$ denote the set of complex $N \times N$ matrices, and let $\mathbb{M}_{d,N}$ denote the set of complex $d \times N$ matrices.) If A does not have a unit diagonal, one can locate two diagonal elements so that $a_{jj} < 1 < a_{kk}$; otherwise, the trace condition would be violated. It is then possible to construct a real plane rotation Q in the jk-plane so that $(Q^* A Q)_{jj} = 1$. The transformation $A \mapsto Q^* A Q$ preserves the conjugate symmetry and the spectrum of A but reduces the number of non-unit diagonal entries by at least one. Thus, at most $(N-1)$ rotations are required before the resulting matrix has a unit diagonal.

The appropriate form of the rotation is easy to discover, but the following derivation is essential to ensure numerical stability. Recall that a two-dimensional plane rotation is an orthogonal matrix of the form

$$Q = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

where $c^2 + s^2 = 1$ [17]. The corresponding plane rotation in the jk-plane is the N-dimensional identity matrix with its jj, jk, kj and kk entries replaced by the entries of the two-dimensional rotation. Let $j < k$ be indices so that

$$a_{jj} < 1 < a_{kk} \qquad \text{or} \qquad a_{kk} < 1 < a_{jj} .$$

The desired plane rotation yields the matrix equation

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix}^* \begin{bmatrix} a_{jj} & a_{jk} \\ a_{jk}^* & a_{kk} \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} 1 & \widetilde{a}_{jk} \\ \widetilde{a}_{jk}^* & \widetilde{a}_{kk} \end{bmatrix}$$

where $c^2 + s^2 = 1$. The equality of the upper-left entries can be stated as

$$c^2 a_{jj} - 2 s c \operatorname{Re} a_{jk} + s^2 a_{kk} = 1 .$$

This equation is quadratic in $t = s/c$:

$$(a_{kk} - 1)\, t^2 - 2 t \operatorname{Re} a_{jk} + (a_{jj} - 1) = 0$$

whence

$$t = \frac{\operatorname{Re} a_{jk} \pm \sqrt{(\operatorname{Re} a_{jk})^2 - (a_{jj} - 1)(a_{kk} - 1)}}{a_{kk} - 1} . \qquad (8)$$

Notice that the choice of j and k guarantees a positive discriminant. As is standard in numerical analysis, the $\pm$ sign in (8) must be taken to avoid cancelations. If necessary, one can extract the other root using the fact that the product of the roots equals $(a_{jj} - 1)/(a_{kk} - 1)$. Finally,

$$c = \frac{1}{\sqrt{1 + t^2}} \qquad \text{and} \qquad s = c t . \qquad (9)$$

Floating-point arithmetic is inexact, so the rotation may not yield $a_{jj} = 1$. A better implementation sets $a_{jj} = 1$ explicitly. Davies and Higham prove that the algorithm is backward stable, so long as it is implemented the way we have described [16]. We restate the algorithm.

Algorithm 1 (Bendel–Mickey): Given Hermitian $A \in \mathbb{M}_N$ with $\operatorname{Tr} A = N$, this algorithm yields a correlation matrix whose eigenvalues are identical with those of A.

1) While some diagonal entry $a_{jj} \neq 1$, repeat Steps 2–4.
2) Find an index k (without loss of generality $j < k$) for which $a_{jj} < 1 < a_{kk}$ or $a_{kk} < 1 < a_{jj}$.

3) Determine a plane rotation Q in the jk-plane using equations (8) and (9).
4) Replace A by Q^*AQ. Set $a_{jj} = 1$.

Since the loop executes no more than $(N-1)$ times, the total cost of the algorithm is no more $12N^2$ real floating-point operations, to highest order, if conjugate symmetry is exploited. The plane rotations never need to be generated explicitly, and all the intermediate matrices are Hermitian. Therefore, the algorithm must store only $N(N+1)/2$ complex floating-point numbers. MATLAB 6 contains a version of Algorithm 1 that starts with a random matrix of specified spectrum. The command is `gallery('randcorr', ...)`.

It should be clear that a similar algorithm can be applied to any Hermitian matrix A to produce another Hermitian matrix with the same spectrum but whose diagonal entries are identically equal to $\operatorname{Tr} A/N$.

The columns of S^* must form an orthogonal basis for the column space of $A \stackrel{\text{def}}{=} S^*S$ according to (6). Therefore, one can use a rank-revealing QR factorization to extract a signature sequence S from the output A of Algorithm 1 [17].

B. Direct Construction of the Signature Matrix

In fact, the methods of the last section can be modified to compute the signature sequence directly without recourse to an additional QR factorization. Any correlation matrix $A \in \mathbb{M}_N$ can be expressed as the product S^*S where $S \in \mathbb{M}_{r,N}$ has columns of unit norm and dimension $r \geq \operatorname{rank} A$. With this factorization, the two-sided transformation $A \mapsto Q^*AQ$ is equivalent to a one-sided transformation $S \mapsto SQ$. In consequence, the machinery of Algorithm 1 requires little adjustment to produce these factors. We have observed that it can also be used to find the factors of an N-dimensional correlation matrix with rank $r < N$, in which case S may take dimensions $d \times N$ for any $d \geq r$.

Algorithm 2 (Davies–Higham): Given $S \in \mathbb{M}_{d,N}$ for which $\operatorname{Tr} S^*S = N$, this procedure yields a $d \times N$ matrix with the same singular values as S but with unit-norm columns.

1) Calculate and store the column norms of S.
2) While some column has norm $\|\boldsymbol{s}_j\|_2^2 \neq 1$, repeat Steps 3–7.
3) Find indices $j < k$ for which
$$\|\boldsymbol{s}_j\|_2^2 < 1 < \|\boldsymbol{s}_k\|_2^2 \qquad \text{or} \qquad \|\boldsymbol{s}_k\|_2^2 < 1 < \|\boldsymbol{s}_j\|_2^2 .$$
4) Form the quantities
$$a_{jj} = \|\boldsymbol{s}_j\|_2^2, \quad a_{jk} = \langle \boldsymbol{s}_k, \boldsymbol{s}_j \rangle \quad \text{and} \quad a_{kk} = \|\boldsymbol{s}_k\|_2^2 .$$
5) Determine a rotation Q in the jk-plane using equations (8) and (9).
6) Replace S by SQ.
7) Update the two column norms that have changed.

Step (1) requires $4dN$ real floating-point operations, and the remaining steps require $12dN$ real floating-point operations to highest order. The algorithm requires the storage of dN complex floating-point numbers and N real numbers for the current column norms. Davies and Higham show that the algorithm is numerically stable [16].

C. Random Unit-Norm Tight Frames

To generate a random signature sequence using the Davies–Higham algorithm, one begins with a matrix S whose d non-zero singular values all equal $\sqrt{N/d}$. There is only one way to build such a matrix: Select for its rows d orthogonal vectors of norm $\sqrt{N/d}$ from $\mathbb{C}^N$. One might choose a favorite orthonormal system from $\mathbb{C}^N$, pick d vectors from it, multiply them by $\sqrt{N/d}$ and use them as the rows of S [13].

Following [16], we can suggest a more general approach. Stewart has demonstrated how to construct a real, orthogonal matrix uniformly at random [18]. Use his technique to choose a random orthogonal matrix; strip off the first d rows; rescale them by $\sqrt{N/d}$; and stack these row vectors to form S. Then apply Algorithm 2 to obtain a unit-norm tight frame. We may view the results as a random unit-norm tight frame (UNTF) [16]. It should be noted that the statistical distribution of the output is unknown [19], although it includes every real UNTF. A version of Algorithm 2 is implemented in MATLAB 6 as `gallery('randcolu', ...)`. An identical procedure using random unitary matrices can be used to construct complex signatures.

IV. CONSTRUCTING WEIGHTED SIGNATURE SEQUENCES

Every optimal weighted signature sequence has a Gram matrix $A \stackrel{\text{def}}{=} X^*X$ with fixed diagonal and spectrum (and conversely). Unfortunately, neither Algorithm 1 not Algorithm 2 can be used to build these matrices. Instead, we must develop a technique for constructing a Hermitian matrix with prescribed diagonal and spectrum. This algorithm, due to Chan and Li, begins with a diagonal matrix of eigenvalues and applies a sequence of plane rotations to impose the power constraints. Our matrix theoretic approach allows us to develop a new one-sided version of the Chan–Li algorithm.

A. A Numerically Stable, Finite Algorithm

Chan and Li present a beautiful, constructive proof of the converse part of the Schur–Horn Theorem [11]. Suppose that $\boldsymbol{w}$ and $\boldsymbol{\lambda}$ are N-dimensional, real vectors for which $\boldsymbol{w} \succcurlyeq \boldsymbol{\lambda}$. Using induction on the dimension, we show how to construct a Hermitian matrix with diagonal $\boldsymbol{w}$ and spectrum $\boldsymbol{\lambda}$. In the sequel, assume without loss of generality that the entries of $\boldsymbol{w}$ and $\boldsymbol{\lambda}$ have been sorted in ascending order. Therefore, $w_{(k)} = w_k$ and $\lambda_{(k)} = \lambda_k$ for each k.

Suppose that $N = 2$. The majorization relation implies $\lambda_1 \leq w_1 \leq w_2 \leq \lambda_2$. Let $A \stackrel{\text{def}}{=} \operatorname{diag} \boldsymbol{\lambda}$. We can explicitly construct a plane rotation Q so that the diagonal of Q^*AQ equals $\boldsymbol{w}$:

$$Q \stackrel{\text{def}}{=} \frac{1}{\sqrt{\lambda_2 - \lambda_1}} \begin{bmatrix} \sqrt{\lambda_2 - w_1} & \sqrt{w_1 - \lambda_1} \\ -\sqrt{w_1 - \lambda_1} & \sqrt{\lambda_2 - w_1} \end{bmatrix}. \tag{10}$$

Since Q is orthogonal, Q^*AQ retains spectrum $\boldsymbol{\lambda}$ but gains diagonal entries $\boldsymbol{w}$.

Suppose that, whenever $\boldsymbol{w} \succcurlyeq \boldsymbol{\lambda}$ for vectors of length $N-1$, we can construct an orthogonal transformation Q so that $Q^*(\operatorname{diag} \boldsymbol{\lambda})Q$ has diagonal entries $\boldsymbol{w}$.

Consider N-dimensional vectors for which $\boldsymbol{w} \succcurlyeq \boldsymbol{\lambda}$. Let $A \stackrel{\text{def}}{=} \operatorname{diag} \boldsymbol{\lambda}$. The majorization condition implies that $\lambda_1 \leq w_1 \leq w_N \leq \lambda_N$, so it is always possible to select a least integer $j > 1$ so that $\lambda_{j-1} \leq w_1 \leq \lambda_j$. Let P_1 be a permutation matrix for which

$$P_1^* A P_1 = \operatorname{diag}(\lambda_1, \lambda_j, \lambda_2, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_N).$$

Observe that $\lambda_1 \leq w_1 \leq \lambda_j$ and $\lambda_1 \leq \lambda_1 + \lambda_j - w_1 \leq \lambda_j$. Thus we may use equation (10), replacing λ_2 with λ_j, to construct a plane rotation Q_2 that sets the first entry of $Q_2^*(\operatorname{diag}(\lambda_1, \lambda_j))Q_2$ to w_1. If we define the rotation

$$P_2 \stackrel{\text{def}}{=} \begin{bmatrix} Q_2 & \mathbf{0}^* \\ \mathbf{0} & \mathrm{I}_{N-2} \end{bmatrix}$$

then

$$P_2^* P_1^* A P_1 P_2 = \begin{bmatrix} w_1 & \boldsymbol{v}^* \\ \boldsymbol{v} & A_{N-1} \end{bmatrix}$$

where $\boldsymbol{v}$ is an appropriate vector and $A_{N-1} = \operatorname{diag}(\lambda_1 + \lambda_j - w_1, \lambda_2, \dots, \lambda_{j-1}, \lambda_{j+1}, \dots, \lambda_N)$.

To apply the induction hypothesis, it remains to check that the vector $(w_2, w_3, \dots, w_N)$ majorizes the diagonal of A_{N-1}. We accomplish this in three steps. First, recall that $\lambda_k \le w_1$ for $k = 2, \dots, j-1$. Therefore,

$$\sum_{k=2}^{m} w_k \ge (m-1)\, w_1 \ge \sum_{k=2}^{m} \lambda_k$$

for each $m = 2, \dots, j-1$. The sum on the right-hand side obviously exceeds the sum of the smallest $(m-1)$ entries of $\operatorname{diag} A_{N-1}$, so the first $(j-2)$ majorization inequalities are in force. Second, use the fact that $\boldsymbol{w} \succcurlyeq \boldsymbol{\lambda}$ to calculate that

$$\begin{aligned}\sum_{k=2}^{m} w_k &= \sum_{k=1}^{m} w_k - w_1 \ge \sum_{k=1}^{m} \lambda_k - w_1 \\ &= (\lambda_1 + \lambda_j - w_1) + \sum_{k=2}^{j-1} \lambda_k + \sum_{k=j+1}^{m} \lambda_k\end{aligned}$$

for $m = j, \dots, N$. Once again, observe that the sum on the right-hand side exceeds the sum of the smallest $(m-1)$ entries of $\operatorname{diag} A_{N-1}$, so the remaining majorization inequalities are in force. Finally, rearranging the relation $\sum_{k=1}^{N} w_k = \sum_{k=1}^{N} \lambda_k$ yields $\sum_{k=2}^{N} w_k = \operatorname{Tr} A_{N-1}$.

In consequence, the induction furnishes a rotation Q_{N-1} which sets the diagonal entries of A_{N-1} equal to the numbers $(w_2, \dots, w_N)$. Define

$$P_3 \overset{\text{def}}{=} \begin{bmatrix} 1 & \mathbf{0}^* \\ \mathbf{0} & Q_{N-1} \end{bmatrix}.$$

Conjugating A by the orthogonal matrix $P = P_1 P_2 P_3$ transforms the diagonal entries of A to $\boldsymbol{w}$ while retaining the spectrum $\boldsymbol{\lambda}$. The proof yields the following algorithm.

Algorithm 3 (Chan–Li): Let $\boldsymbol{w}$ and $\boldsymbol{\lambda}$ be vectors with ascending entries and such that $\boldsymbol{w} \succcurlyeq \boldsymbol{\lambda}$. The following procedure computes a real, symmetric matrix with diagonal entries $\boldsymbol{w}$ and eigenvalues $\boldsymbol{\lambda}$.

1) Initialize $A = \operatorname{diag} \boldsymbol{\lambda}$, and put $n = 1$.
2) Find the least $j > n$ so that $a_{j-1,j-1} \le w_n \le a_{jj}$.
3) Use a symmetric permutation to set $a_{n+1,n+1}$ equal to a_{jj} while shifting diagonal entries $n+1, \dots, j-1$ one place down the diagonal.
4) Define a rotation Q in the $(n, n+1)$-plane with
$$c = \sqrt{\frac{a_{n+1,n+1} - w_n}{a_{n+1,n+1} - a_{nn}}}, \quad s = \sqrt{\frac{w_n - a_{nn}}{a_{n+1,n+1} - a_{nn}}}.$$
5) Replace A by $Q^* A Q$.
6) Use a symmetric permutation to re-sort the diagonal entries of A in ascending order.
7) Increment n, and repeat Steps 2–7 while $n < N$.

This algorithm requires about $6N^2$ real floating-point operations. It requires the storage of about $N(N+1)/2$ real floating-point numbers, including the vector $\boldsymbol{w}$. It is conceptually simpler to perform the permutations described in the algorithm, but it can be implemented without them.

We have observed that the algorithm given by Viswanath and Anantharam [2] for constructing gWBEs is identical with Algorithm 3.

B. A New One-Sided Algorithm

Algorithm 3 only produces a Gram matrix, which must be factored to obtain the weighted signature matrix. We propose a new one-sided version. The benefits are several. It requires far less storage and computation than the Chan–Li algorithm. At the same time, it constructs the factors explicitly.

Algorithm 4: Suppose that $\boldsymbol{w}$ and $\boldsymbol{\lambda}$ are non-negative vectors of length N with ascending entries. Assume, moreover, that the first $(N-d)$ components of $\boldsymbol{\lambda}$ are zero and that $\boldsymbol{w} \succcurlyeq \boldsymbol{\lambda}$. The following algorithm produces a $d \times N$ matrix X whose column norms are listed by $\boldsymbol{w}$ and whose squared singular values are listed by $\boldsymbol{\lambda}$.

1) Initialize $n = 1$, and set
$$X = \left[\begin{array}{c|ccc} \mathbf{0} & \sqrt{\lambda_{N-d+1}} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_N} \end{array}\right].$$
2) Find the least $j > n$ so that $\|\boldsymbol{x}_{j-1}\|_2^2 \le w_n \le \|\boldsymbol{x}_j\|_2^2$.
3) Move the j-th column of X to the $(n+1)$-st column, shifting the displaced columns to the right.
4) Define a rotation Q in the $(n, n+1)$-plane with
$$c = \sqrt{\frac{\|\boldsymbol{x}_{n+1}\|_2^2 - w_n}{\|\boldsymbol{x}_{n+1}\|_2^2 - \|\boldsymbol{x}_n\|_2^2}}, \quad s = \sqrt{\frac{w_n - \|\boldsymbol{x}_n\|_2^2}{\|\boldsymbol{x}_{n+1}\|_2^2 - \|\boldsymbol{x}_n\|_2^2}}.$$
5) Replace X by XQ.
6) Sort columns $(n+1), \dots, N$ in order of increasing norm.
7) Increment n, and repeat Steps 2–7 while $n < N$.

Note that the algorithm can be implemented without permutations. The computation requires $3dN$ real floating-point operations and storage of $N(d+2)$ real floating-point numbers including the desired column norms and the current column norms. This is far superior to the other algorithms outlined here, and it also bests the algorithms from the information theory literature. Moreover, the algorithm is numerically stable because the rotations are properly calculated.

V. Conclusions and Further Work

We have discussed a group of four algorithms that can be used to produce sum-capacity-optimal S-CDMA sequences in a wide variety of circumstances. Algorithm 1 constructs a Hermitian matrix with a constant diagonal and a prescribed spectrum. This matrix can be factored to yield an optimal signature sequence for the case of equal user powers, i.e. a unit-norm tight frame. Alternately, Algorithm 2 can be used to produce the factors directly. In constrast, Algorithm 3 constructs a Hermitian matrix with an arbitrary diagonal and prescribed spectrum, subject to the majorization condition. The resulting matrix can be factored to obtain an optimal signature sequence for the case of unequal received powers, i.e. a tight frame. We have also introduced an efficient new variant, Algorithm 4, that can calculate the factors directly.

Algorithms 1 and 2 can potentially calculate every correlation matrix and its factors. If they are initialized with random matrices, one may interpret the output as a random correlation matrix. The factors can be interpreted as random unit-norm signature sequences.

On the other hand, the output of Algorithms 3 and 4 is not encyclopedic. They can construct only a few matrices for each pair $(\boldsymbol{w}, \boldsymbol{\lambda})$. These matrices are also likely to have many zero entries, which is undesirable for some applications. In addition, these algorithms only build real matrices, whereas complex matrices are often of more interest.

One may observe that Algorithms 1 and 3 always change the diagonal in the $\succcurlyeq$-increasing direction. Using this insight, we have developed generalizations of both algorithms. For more details, refer to [20].

Matrix analysis can provide powerful tools for solving related sequence design problems. For example, we have developed an iterative technique that can compute optimal signature sequence which satisfy additional constraints, such as unimodularity of the components [8]. Related methods can even construct Maximum Welch-Bound-Equality sequences (MWBEs), which is a more challenging problem [21].

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